Received: *March 3, 2000*, Accepted: *April 4, 2000*

Lattice calculation of $1/p^2$ corrections to α_s and of Λ_{QCD} in the $\widetilde{MOM}$ scheme.

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Abstract: We report on very strong evidence of the occurrence of power terms in $\alpha_{\widetilde{\mathrm{MOM}}}(p)$, the QCD running coupling constant in the $\widetilde{MOM}$ scheme, by analyzing non-perturbative measurements from the lattice three-gluon vertex between 2.0 and 10.0 GeV at zero flavor. While putting forward the caveat that this definition of the coupling is a gauge dependent one, the general relevance of such an occurrence is discussed. We fit $\Lambda_{\overline{\mathrm{MS}}}^{(n_f=0)} = 237 \pm 3\,^{+\,0}_{-10}\,\mathrm{MeV}$ in perfect agreement with the result obtained by the ALPHA group with a totally different method. The power correction to $\alpha_{\widetilde{\mathrm{MOM}}}(p)$ is fitted to $(0.63 \pm 0.03\,^{+\,0.0}_{-\,0.13})\,\mathrm{GeV}^2/p^2$.

Keywords: Renormalization Regularization and Renormalons, Nonperturbative Effects, QCD, Lattice QCD.

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Contents

1. Introduction

The non-perturbative computation of the running QCD coupling constant $\alpha_s(p)$ follows a two-sided goal: the large energy matching to perturbative asymptotic QCD formula turns out to be a most direct method to predict $\Lambda_{\overline{\rm MS}}$ from QCD first principles [1]. On the other hand, the moderate or low energy behavior of $\alpha_s(p)$ is extremely instructive about non-perturbative properties of QCD. In this paper we restrict ourselves to high and intermediate energies and consider power corrections ($\sim 1/p^2$) to the leading asymptotic behavior. As we shall see it turns out that there is no sharp separation between the asymptotic domain and the intermediate one. The power correction, beyond the lessons it contains by itself, greatly improves our asymptotic study and is never negligible up to $\sim 10\,$GeV! This surprising fact could only be revealed thanks to the high accuracy achieved in the present work.

The asymptotic approach has been recently followed in ref. [2]. It is worth remarking that this matching procedure has also been developed for the two-point Green function to the same goal [3], both matching programs leading to a consistent estimate of $\Lambda_{\overline{\rm MS}}$. The almost two-sigma discrepancy between the last estimate and the one obtained from Schrödinger functional methods [4] seems to imply that some source of systematic uncertainty remains uncontrolled. Furthermore the careful study of the asymptotic behavior carried out for the gluon propagator in refs. [3] stresses a danger: it is sometimes difficult to distinguish between the *real asymptotic scaling* and a behavior *mimicking asymptoticity* but with a certain effective "re-scaled" Λ parameter which differs significantly from the real asymptotic one. Consequently, and in spite of the agreement between both above-mentioned estimates of $\Lambda_{\overline{\mathrm{MS}}}$ (matching two or three-point Green functions to perturbative formulae), we must inquire whether both are not biased by some sizable non-perturbative effect.

The operator product expansion (OPE) gives a standard procedure to parametrize non-perturbative QCD effects in terms of power corrections to perturbative results. In this framework, the powers involved in the expansion are expected to be uniquely fixed by the symmetries and the dimensions of the operators appearing in the product expansion. It should be noted that, due to the asymptotic nature of QCD perturbative expansions, power corrections are reshuffled between operators and coefficient functions in the OPE [5].

Since we work in Landau gauge, the gauge dependent local operator $A_\mu A^\mu$ is allowed in the OPE [6] implying a dominant $1/p^2$ power correction. Indeed sizable $1/p^2$ corrections are present as we shall show at length in the next section.

In a less straightforward way our result is related to another hot issue (this is the spirit of the preliminary study in [7]): the possible presence of $1/p^2$ terms in *gauge invariant* quantities such as Wilson loops [8]. Since no gluon local gauge invariant operator of dimension less than 4 exists it is expected from OPE that the dominant power correction should be $\propto 1/p^4$. This picture has however recently been challenged [9, 10, 8]. It was pointed out that power corrections which are not *a priori* expected from the OPE may in fact appear in the expansion of physical observables. Such terms may arise from (UV-subleading) power corrections to $\alpha_s(p)$, corresponding to non-analytical contributions to the β-function. It is worth stressing the following. One knows that the perturbative analysis does not encode all the information on the coupling. Among all that is missed, a *peculiar* contribution to the coupling could result in a *peculiar* correction to physical observables. As a matter of fact, some evidence for an unexpected Λ^2/p^2 contribution to the gluon condensate was obtained through lattice calculations in ref. [8] (see also [11] for an early evidence of such a contribution, although the perturbative series involved was not managed up to high orders).

Let us insist: there is no *direct* relation between the $1/p^2$ corrections found in the present work to $\alpha_s(p)$ in a *gauge dependent scheme* and the power corrections advocated in the preceding paragraph to justify $1/p^2$ corrections to *gauge independent quantities*. Nobody knows how to relate a gauge dependent scheme to a gauge independent one beyond the perturbative regime. Still, it may be that these two phenomena are not completely unrelated and the large $1/p^2$ corrections found in the present work might trigger some further thoughts along this line.

The paper is organized as follows: in section 2 we explain the meaning of the lattice data, our strategy for the analysis and report the results. In section 3 we briefly review some theoretical arguments in support of power corrections to $\alpha_s(p)$, illustrating the special role of the Λ^2/p^2 term. Finally, in section 4 we draw our conclusions.

2. Lattice calculation of α_s from Green functions and power corrections

2.1 α_s on the lattice

Several methods for computing $\alpha_s(p)$ non-perturbatively on the lattice have been proposed in recent years [12]–[16]. In most cases, the goal of such studies is to obtain an accurate prediction for $\alpha_s(M_Z)$, i.e. the running coupling at the Z peak, which is a fundamental parameter in the standard model. For this reason, lattice parameters are usually tuned so as to allow the computation of $\alpha_s(p)$ at momentum scales of at least a few GeVs, where the two-loop asymptotic behavior is expected to dominate and power contributions are suppressed.

As we shall see, however, non-perturbative power corrections cannot be neglected at energy scales as large as 10 GeV which is a sufficient reason to consider them in the fit. As a bonus the knowledge of these power corrections provides us with a physically significant quantity as argued in the introduction.

For this purpose, the best method is one where one can measure $\alpha_s(p)$ in a wide range of momenta from a single Monte Carlo data set. Indeed, a narrow energy window does not allow to disentangle in a clear cut manner the power corrections from unknown higher perturbative orders, and these corrections can be mimicked by an effective $\Lambda_{\rm QCD}$ different from the asymptotic one.

One method which fulfills the above criterion is the determination of the coupling from the renormalized lattice three-gluon vertex function [16, 1, 2]. This is achieved by evaluating two- and three-point off-shell Green's functions of the gluon field in the Landau gauge, and imposing non-perturbative renormalization conditions on them, for different values of the external momenta. By varying the renormalization scale p, one can determine $\alpha_s(p)$ for different momenta from a single simulation. Obviously the renormalization scale must be chosen in a range of lattice momenta such that both finite volume effects and discretization errors are under control. Both these constraints impose too strict limits on the energy range if only one lattice run is used. This is why the procedure used in ref. [3] combining several lattice simulations at different β's is a necessity to get a larger range of lattice momenta. The use of different volumes will also help to control finite volume artifacts.

The definition of the coupling corresponds to a momentum-subtraction renormalization scheme in continuum QCD [17]. It should be noted that in this scheme the coupling is a gauge-dependent quantity. As we already mentioned, one consequence of this fact is that $1/p^2$ corrections should be expected, based on OPE considerations. For full details of the method and the lattice calibration we refer the reader to ref. [1, 2].

2.2 Models for power corrections and construction of the data set

In the present work we shall not address the general problem of defining an optimal analytic form for the coupling at all scales to which we could fit our data. For the purpose of our investigation, we shall compare the non-perturbative data for α_s with simple models obtained by adding a power correction term of the form $1/p^2$ to the perturbative formula at a given order. This amounts to a quite raw separation between *a perturbative versus a non-perturbative contribution*, the major problem of course being the possible interplay between a description in terms of (non-perturbative) power corrections and our ignorance about higher orders of perturbation theory. As crude as it may be, our recipe allows for addressing this problem, as we shall see.

In order to identify a momentum interval where our ansatz could fit the data, one should keep in mind that the momentum range should start well above the location of the perturbative Landau pole, but it should nonetheless include low scales where power corrections may be large up to large enough momentum scales in order to be confident that the asymptotic regime (i.e. perturbative running) has become dominant. Our choice will be[1] $2.0\,\mathrm{GeV} \leq p \leq 9.6\,\mathrm{GeV}$. It will turn out that in the *whole range* both the perturbative three-loop contribution and the non perturbative $1/p^2$ one contribute by a sizable amount, although obviously the former becomes dominant at larger scales.

A data set which fulfills the above-mentioned requirements can be constructed by aggregating data computed at different β values ($\beta = 6.0, 6.2, 6.4, 6.8$) on a 24^4 lattice. The fact that such a data set can be assembled is a very good *a posteriori* check that the expected scaling in the lattice cutoff a takes place.

On the other hand, the physical volumes for these simulations are very different from each other. The appropriate matching of the data proves that finite-size effects remain controlled. The pattern for these volume effects is clear from figure 1a, where the whole set of data is plotted, including the points too much affected by finite volume artifacts and eventually rejected in our fits. As can be seen, the effects are negligible for large enough Lp, where L is the physical lattice length. Evidences for the same trend arise from the comparison of data obtained on two different lattices (16^4, 24^4) at $\beta = 6.8$, shown in figure 1b. In practice we will take as infrared cut-off at each β the value $p_{\min}$ such that, including points below this value, the $\chi^2_{\mathrm{d.o.f.}}$ increases dramatically. This criterion leads to: $p_{\min}(6.0) = 2.0\,\mathrm{GeV}$, $p_{\min}(6.2) = 2.5\,\mathrm{GeV}$,

[1] The range's upper limit is determined by the condition $a, p \leq \pi/2$, where a is the lattice spacing, applied to our 24^4 lattice at $\beta = 6.8$.

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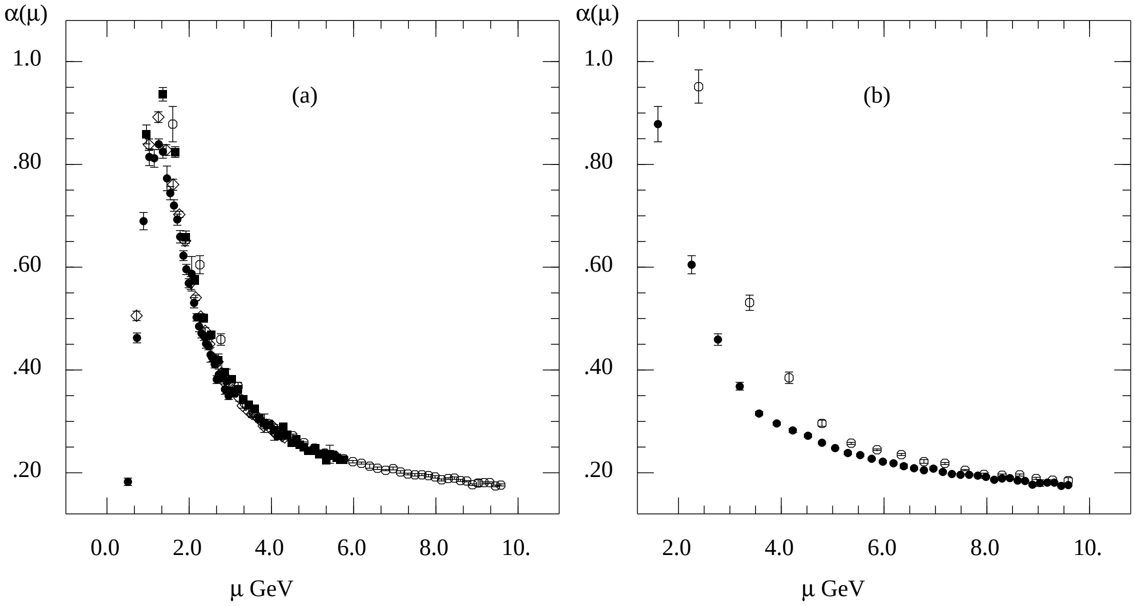

Figure 1: Evaluations of α_S from 24^4 lattices at $\beta = 6.0, 6.2, 6.4, 6.8$ are respectively shown in plot (a) with black circles, white diamonds, black squares and white circles. In plot (b) black (white) circles correspond to α_S evaluations from a 24^4 (16^4) lattice at $\beta = 6.8$.

$p_{\min}(6.4) = 4.0\,\mathrm{GeV}$ and $p_{\min}(6.8) = 6.0\,\mathrm{GeV}$. It corresponds to $Lp \gtrsim 24$.[2] Reversely, when we vary the infrared cut-offs above the values just mentioned, the $\chi^2_{\text{d.o.f.}}$ stays stable, in the range 1.4 to 1.6. Even more striking, the resulting value for $\Lambda_{\overline{\rm MS}}$ is very stable, varying no more than 2 MeV. Consequently the data set obtained with these cut-offs should be considered as IR safe and will be used in the following fits.

At each β value one should of course also worry of data in the range of the larger values of momentum, which are affected by lattice artifacts of $O(a^2p^2)$ (UV discretization effects). This was taken care of by the *sinus-improvement* program which has already been described in [2], which basically amounts to the substitution of the lattice momenta $2\pi n_\mu/L$ with their $O(a^2p^2)$ analogues $\frac{2}{a}\sin(ap_\mu/2)$. As already noticed in [2], a good rationale for this in our gauge-fixed situation is that the gauge fixing algorithm leads to the relation $\frac{2}{a}\sin(ap_\mu/2)\,A_\mu(p) = 0$, while $p_\mu\,A_\mu$ does not vanish.[3] It should be noticed that without this prescription the quality of almost any fit degenerates. The relevant configurations were generated on a QH1 Quadrics system based in Orsay (see [2]).

2.3 Fitting data to our ansatz

We now proceed to fitting the data set to our ansatz, which, we recall, is the addition

[2]Almost all the data from our 16^4 lattice at $\beta = 6.8$ turn out to be excluded by such a requirement, $Lp \gtrsim 24$ leading for example to an infra-red cut-off equal to 9 GeV. This is why we will not use at all the volume 16^4.

[3]One should keep in mind that we are imposing Landau gauge.

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of a term proportional to $1/p^2$ to a given order of perturbation theory:

$$\alpha_s(p^2) = \alpha_s^{\rm Pert}(p^2) + \frac{c}{p^2}\,. \tag{2.1}$$

Working at three loop level, the perturbative expression for the running coupling constant, $\alpha_S^{\rm Pert}(p^2)$ requires to inverse either the unexpanded formula

$$\begin{aligned}\widetilde{\Lambda} = \widetilde{\Lambda}^{(c)}(\widetilde{\alpha}) &\left(1 + \frac{\beta_1 \widetilde{\alpha}}{2\pi\beta_0} + \frac{\widetilde{\beta}_2 \widetilde{\alpha}^2}{32\pi^2\beta_0}\right)^{\beta_1/2\beta_0^2} \times \\ \times \exp &\left\{\frac{\beta_0\widetilde{\beta}_2 - 4\beta_1^2}{2\beta_0^2\sqrt{\Delta}}\left[\arctan\left(\frac{\sqrt{\Delta}}{2\beta_1 + \widetilde{\beta}_2\widetilde{\alpha}/4\pi}\right) - \arctan\left(\frac{\sqrt{\Delta}}{2\beta_1}\right)\right]\right\}\end{aligned} \tag{2.2}$$

or the expanded one

$$\widetilde{\Lambda} = \widetilde{\Lambda}^{(c)}(\widetilde{\alpha})\left(1 + \frac{8\beta_1^2 - \beta_0\widetilde{\beta}_2}{16\pi^2\beta_0^3}\widetilde{\alpha}\right), \tag{2.3}$$

where $\widetilde{\Lambda}^{(c)}$ denotes the conventional two loops formula:

$$\widetilde{\Lambda}^{(c)} \equiv p\exp\left(\frac{-2\pi}{\beta_0\widetilde{\alpha}(p^2)}\right) \times \left(\frac{\beta_0\widetilde{\alpha}(p^2)}{4\pi}\right)^{-\beta_1\beta_0^2}; \tag{2.4}$$

and $\Delta \equiv 2\beta_0\widetilde{\beta}_2 - 4\beta_1^2 > 0$ (for our $\widetilde{\rm MOM}$ scheme). In the previous formula, the use of $\widetilde{\Lambda}$, $\widetilde{\alpha}$ and $\widetilde{\beta}$'s stands for the Λ parameter, the running coupling constant and beta function coefficients in the particular $\widetilde{\rm MOM}$ renormalization scheme. From now on we will systematically convert $\widetilde{\Lambda}$ into $\Lambda_{\overline{\rm MS}}$ using [1, 2]

$$\Lambda_{\overline{\rm MS}} = \exp(-70/66)\,\widetilde{\Lambda} \simeq 0.346\,\widetilde{\Lambda}\,. \tag{2.5}$$

In eqs. (2.2)–(2.3) the p^2 dependence of $\widetilde{\alpha}$ has been omitted for simplicity.

No analytical expression can exactly inverse neither unexpanded eq. (2.2) nor expanded eq. (2.3). The following formula gives an approximated solution to the inversion of eq. (2.3):

$$\widetilde{\alpha}(p^2) = \frac{4\pi}{\beta_0\, t} - \frac{8\pi\beta_1}{\beta_0}\frac{\log(t)}{(\beta_0\, t)^2} + \frac{1}{(\beta_0\, t)^3}\left(\frac{2\pi\widetilde{\beta}_2}{\beta_0} + \frac{16\pi\beta_1^2}{\beta_0^2}(\log^2(t) - \log(t) - 1)\right), \tag{2.6}$$

where $t = \log(p^2/\widetilde{\Lambda}^2)$. On the other hand, an exact numerical inversion of eq. (2.2) can be easily obtained and used to fit our data. Both ansätze, eq. (2.6) and the numerical inversion of eq. (2.2), should only differ by perturbative contributions higher than three loops. Thus, we will fit with the two ansätze, the difference between these two predictions being considered as an estimate of the systematic uncertainty coming

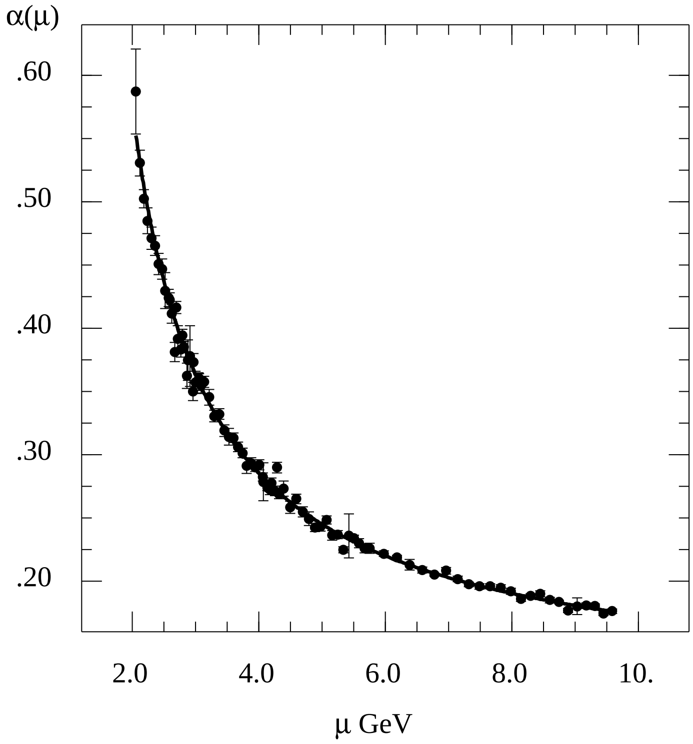

Figure 2: Formula (2.1), with the values of $\Lambda_{\overline{\rm MS}}$ and c given by eq. (2.7), fits impressively the data set built as explained in the text.

from the neglected perturbative orders. To make more direct the comparison with the Schrödinger functional estimate of $\Lambda_{\overline{\rm MS}}$ [4], the central value for our prediction of $\Lambda_{\overline{\rm MS}}$ should be taken from the fit with the exact inversion of eq. (2.2).[4] This yields:

$$\Lambda_{\overline{\rm MS}} = 237 \pm 3\,\text{MeV}\,, \qquad c = 0.63 \pm 0.03\,\text{GeV}^2\,, \qquad \chi^2/\text{d.o.f.} = 1.6 \qquad (2.7)$$

in perfect agreement with the determination [4] which uses a totally different technique. Using eq. (2.6) we obtain $\Lambda_{\overline{\rm MS}} = 227(5)$ MeV and $c = 0.50(6)\,\text{GeV}^2$. Comparing the latter results with eq. (2.7) provides an estimate of the higher loop uncertainty of about 10 MeV for $\Lambda_{\overline{\rm MS}}$ and $0.1\,\text{GeV}^2$ for c. The size of the power correction will be discussed in sections 3.1 and 3.2.

Figure 3 illustrates the effect of the power correction in a striking manner. The upper set of points shows Λ, converted to $\Lambda_{\overline{\rm MS}}$, computed through eq. (2.2) from α_s provided by the lattice at every value p. Scaling would imply a constancy of $\Lambda_{\overline{\rm MS}}$ which is far from true. The lower set of points corresponds to the same formula applied to $\alpha_s(p^2)^{\rm lattice} - 0.63/p^2$, i.e. to what should fit the perturbative formula. Now the constancy of $\Lambda_{\overline{\rm MS}}$ is very good within the statistical errors. It is now clear why the upper points show a trend to decrease: as the energy scale increases, the

[4]The ALPHA collaboration estimate: $\Lambda_{\overline{\rm MS}} = 238(19)$ MeV comes from evaluating eq. (2.2) for a value of α_S obtained from the lattice at very high momenta.

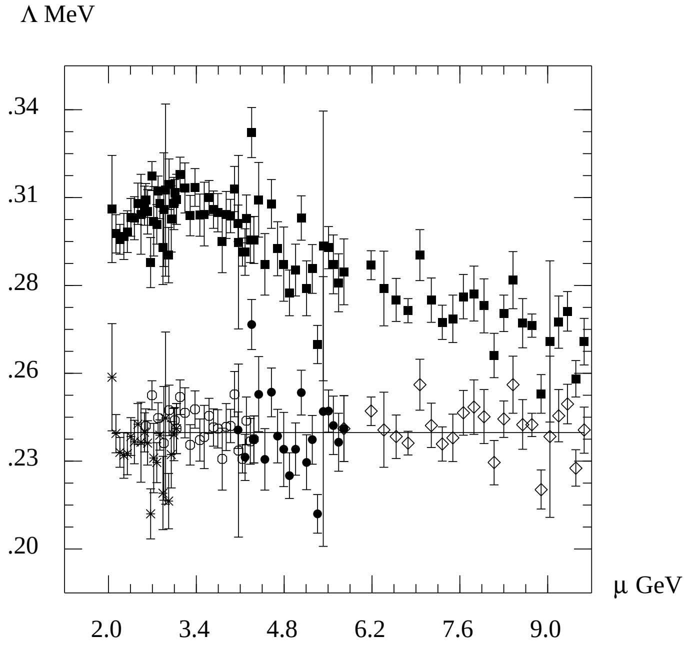

Figure 3: The estimates of $\Lambda_{\overline{\rm MS}}$ computed from α_s via eq. (2.2) are plotted as a function of the renormalization point momenta. Diamonds, black and white circles, and asterisks respectively correspond to evaluations from 24^4 lattices at $\beta = 6.8, 6.4, 6.2, 6.0$. Upper points are computed from the α_s values directly obtained from the lattice, while the lower points use $\alpha_s^{\rm Pert}$ as defined in eq. (2.1) with c given by eq. (2.7).

effect of the power corrections must decrease, and the upper points converge slowly towards the lower ones. The surprise is that above 9.0 GeV the merging of the two sets of points has not yet taken place contrarily to the general expectation that power corrections are negligible at such a scale. We will elaborate on this in the next section.

We note that using eq. (2.6) one can draw the same conclusion: imposing $c = 0$ (i.e. fitting to a pure three loop formula) there is no good fit on the whole range of momenta and the best that one can obtain on a restricted interval yields a value for $\Lambda_{\overline{\rm MS}}$ higher than expected (e.g. with respect to [4]) and a definitly worse $\chi^2_{\rm d.o.f.}$.

We now proceed to address the question of the stability of our previous result with respect to the inclusion of the fourth loop. To fit the value of Λ, one would need β_3, the four loop coefficient in the perturbative expansion of the β-function. This coefficient is unknown but one can pin down a reasonable range of variation. Let us consider the ratio b_3/b_2 with $b_3 = \beta_3/(4\pi)^3$ and $b_2 = \beta_2/(2\pi)^2$. This ratio is larger than one in the only scheme ($\overline{\rm MS}$) for which it is known. On the other hand we expect that, for a reasonably small value of the coupling, the perturbative expansion of the β-function is still in the regime in which it seems to converge at four loops. As a working hypothesis, we then suppose that the contribution to the β-function

b_3/b_2	4 loops				4 loops + power					
	whole		> 3		whole			> 3		
	$\chi^2_{\rm dof}$	$\Lambda_{\overline{\rm MS}}$	$\chi^2_{\rm dof}$	$\Lambda_{\overline{\rm MS}}$	$\chi^2_{\rm dof}$	$\Lambda_{\overline{\rm MS}}$	c	$\chi^2_{\rm dof}$	$\Lambda_{\overline{\rm MS}}$	c
0	7.8	299	8.3	294	1.6	237	0.63	1.6	235	0.67
1	7.1	284	6.0	287	1.8	238	0.56	1.6	235	0.61
2	28	259	4.1	279	2.7	229	0.57	1.6	236	0.53
3	104	227	3.0	270	4.7	215	0.65	1.6	238	0.45
4	145	215	2.9	261	7.3	202	0.75	1.8	237	0.37
5	157	209	4.4	252	10.2	190	0.84	2.4	231	0.37

Table 1: A collection of fitted parameters obtained by imposing different values for the ratio b_3/b_2 defined in the text. "Whole" refers to the whole energy window ($2\,\mathrm{GeV} < \mu < 10\,\mathrm{GeV}$) and "$> 3$" to a momentum range above 3 GeV.

coming from β_3 (that is, the one proportional to α^5) is not too much larger than the one coming from β_2 at a typical value of $\alpha \sim 0.4$, implying $b_3/b_2 < 2.5$. Actually we conservatively vary b_3 from 0 up to $5b_2$ in the following exploration of a large range of values for b_3.

First of all we try a pure four loop fit (that is without any power correction). We observe that there is no good fit on the whole range of momenta. If one tries to add again a term proportional to $1/p^2$ to the four loop perturbative expression, the following should be noted. Good fits are recovered either on the whole range of momenta for $b_3/b_2 \lesssim 1$ or by discarding momenta below 3 GeV as $b_3/b_2 \lesssim 4$. As far as the value of Λ is concerned, it is really stable when the fits are of good quality.[5] As for the coefficient of the non-perturbative term $1/p^2$, it is less stable. Results are summarized in table 1. One can see how the "new player on the ground", β_3, is strongly correlated to the coefficient of the power term. This coefficient is anyhow fully consistent with the value determined at three loop level, as long as $b_3/b_2 \lesssim 2$ i.e. when the asymptotic behavior is not too dramatically perturbed by the four-loop contribution.

Again, much the same holds when fitting to a formula for α_s as a function of momentum (like eq. (2.6)) at the four loop level. In order to trust a four loop more than power corrections one would need both to discard lower momenta and to accept excedingly large values for b_3.

Of course, the perturbative knowledge of β_3 coefficient is unavoidable to get total confidence on our results *versus* higher loop orders inclusion. One should take this last analysis only as a preliminary check. Still it is another indication that things are working pretty well (that is, consistently) with respect to our theoretical prejudice. In any case, the value obtained for $\Lambda_{\overline{\rm MS}}$ is almost insensitive to β_3.

[5] Even for $b_3/b_2 = 5$, one could obtain a good quality fit ($\chi^2 = 1.8$) with $\Lambda_{\overline{\rm MS}} = 238$, if momenta below 4 GeV are now discarded.

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We collect all the hints from these many counterproofs. To estimate the systematic errors we use two methods: first we compare the exact inversion of eq. (2.2) and the use of eq. (2.6) on our results, second we use the results on the whole energy range in table 1 with a reasonable χ^2. We take the three loop result with formula (2.2) as our central value

$$\Lambda_{\overline{\rm MS}} = 237 \pm 3 \,^{+\;0}_{-10}\,\mathrm{MeV}\,, \qquad c = 0.63 \pm 0.03 \,^{+\;0.0}_{-\;0.13}\,\mathrm{GeV}^2 \tag{2.8}$$

The present analysis has improved over the one presented in [1, 2] by using a larger data set which provides a wider momentum window and taking into account a power correction. If one tried to repeat the fits only in the range of the lower momenta as we did in [1, 2], there would be no clear cut indication for power corrections and the effective value for Λ would turn out to be higher. What the upper momenta data really do is to single out the asymptotic value for Λ, while deviation from asymptotia in the lower momenta data asks for power correction rather than higher loops.

3. Discussion on the $1/p^2$ corrections to $\alpha_s(p)$

Power corrections to $\alpha_s(p)$ can be shown to arise naturally in many physical schemes [18, 19]. The occurrence of such corrections cannot be excluded *a priori* in any renormalization scheme. Even more so, as previously stated, in a gauge dependent renormalization scheme as the $\widetilde{\rm MOM}$ discussed here. Clearly, the non-perturbative nature of such effects makes it very hard to assess their dependence on the renormalization scheme, which is only very weakly constrained by the general properties of the theory.

As discussed in the following, several arguments have been put forward in the past to suggest that a most likely candidate for a leading power correction to $\alpha_s(p)$ would be the same term of order Λ^2/p^2 we found. Furthermore, it is worth emphasizing that this does not contradict the OPE expectation for a gauge dependent quantity.

3.1 Static quark potential and confinement

Consider the interaction of two heavy quarks in the static limit (for a more detailed discussion and a recent review of the subject see [20]). In the one-gluon-exchange approximation, the static coulombic potential $V(r)$ can be written as

$$V(r) \propto \alpha_s \int d^3k \, \frac{\exp^{i\vec{k}\cdot\vec{r}}}{|\vec{k}|^2}\,. \tag{3.1}$$

Using standard arguments of renormalon analysis, one may consider a generalization of (3.1) obtained by replacing α_s with a running coupling:

$$V(r) \propto \int d^3k \alpha_s(|\vec{k}|^2)\, \frac{\exp^{i\vec{k}\cdot\vec{r}}}{|\vec{k}|^2}\,. \tag{3.2}$$

The presence in $\alpha_s(k^2)$ of a power correction term of the form c/k^2 would generate a linear confining piece Kr in the potential $V(r)$. It would of course be totally unjustified to take seriously our power correction in order to derive the linear potential. If we nevertheless perform this exercise to have a feeling of the scales, we get $K = 2/3c \simeq 0.4\,\mathrm{GeV}^2$, while the usual string tension is around $0.2\,\mathrm{GeV}^2$. Note that a standard renormalon analysis of (3.2) (see [20] for the details) reveals contributions to the potential containing various powers of r, but a linear contribution is missing. This is a typical result of renormalon analysis: renormalons can miss important pieces of non-perturbative information.

3.2 An estimate from another lattice method

The lattice community has been made aware for some time of the possibility of non-perturbative contributions to the running coupling; for a clear discussion see [21]. Consider the "force" definition of the running coupling:

$$\alpha_{q\bar{q}}(Q) = \frac{3}{4} r^2 \frac{dV(r)}{dr}\,, \qquad \left(Q = \frac{1}{r}\right), \tag{3.3}$$

where again $V(r)$ represents the static interquark potential. By keeping into account the string tension contribution to $V(r)$, which can be measured in lattice simulations, one obtains a $1/Q^2$ contribution, whose order of magnitude is given by the string tension itself. Ironically, this term has been mainly considered as a sort of ambiguity, resulting in an indetermination in the value of $\alpha(Q)$ at a given scale. From a different point of view, such a term could be interpreted as a clue for the existence of a Λ^2/p^2 contribution, and it also provides an estimate for the expected order of magnitude of it, at least in one (physically sound) scheme. The same naïve exercise than in the preceding subsection leads from eq. (3.3) to $K = 4/3c \simeq 0.8\,\mathrm{GeV}^2$.

In order to make a deeper contact with what we are actually studying, one should of course keep in mind that all the preceding arguments would be physically sounder in the Coulomb gauge in which the notion of force has a clearer meaning. In Landau gauge such a picture is far from clear. Furthermore, it is known that the interquark potential becomes close to linear at distances larger than 0.5 fm, which corresponds to a momentum smaller than 0.4 GeV, while our $1/p^2$ fit only apply above 2.0 GeV i.e. at distances smaller than 0.1 fm. Still, a couple of considerations are in order at this point. We note that the rough order of magnitude of the power term we found is the same as inferred from the simple arguments from the static potential (some $10^{-1}\,\mathrm{GeV}^2$), even if we are in a different (UV) regime. While this could turn out to be accidental, it is nevertheless intriguing to think of some sort of relation. In any case, one should nevertheless note that the power correction that we have obtained is rather large with respect to the common wisdom of non-perturbative effects being negligible at scales such as 10.0 GeV. A further study of the relation between the power correction and the confining potential is clearly needed.

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3.3 On the relation with the lattice gluon condensate puzzle.

Having just stressed that the contribution of the power correction is rather large also at a scale such as 10.0 GeV, this is a good point to go back to the argument referring to the unexpected result of [8]. As already mentioned in the introduction, non-perturbative contribution to the running coupling can be advocated in order to explain the Λ^2/Q^2 contribution to the gluon condensate. The argument runs as follows (see [8]). From general arguments one expects the condensate W to be written in the form

$$\int_0^{Q^2} \frac{p^2 dp^2}{Q^4} f\left(\frac{p^2}{\Lambda^2}\right), \tag{3.4}$$

which is based on the fact that this condensate has dimension four and is renormalization group invariant, so that the function $f(p^2/\Lambda^2)$, which is independent on Q (for large Q), can be expressed as a function of a running coupling. This leads to consider the contribution coming from the large frequency part of the integral

$$\int_{\rho\Lambda^2}^{Q^2} \frac{p^2 dp^2}{Q^4} \alpha_s\left(\frac{p^2}{\Lambda^2}\right), \tag{3.5}$$

in which the function $f(p^2/\Lambda^2)$ is taken proportional to the running coupling.[6] By taking into account the perturbative running coupling the IR renormalon contribution can be obtained (see [8] for details). Again, one can consider also contributions coming from a non-perturbative correction to the coupling of the form c/p^2. From the UV limit of integration one then obtains a Λ^2/Q^2 contribution to W. Let us insist: this is a contribution coming from a coupling with a c/p^2 correction in the UV region. While stressing again that one can draw no definite conclusion from the following consideration, still it is worth noting that our scheme provides an example of a coupling in which a c/p^2 correction is not negligible even at 10.0 GeV. For a discussion of possible scenarios for the result of [8] see also [22].

3.4 Landau pole and analyticity.

It is well known that perturbative QCD formulae for the running of α_s inevitably contain singularities, which are often referred to as the Landau pole. The details of the analytical structure depend on the order at which the β-function is truncated and on the particular solution chosen. The existence of an interplay between the analytical structure of the perturbative solution and the structure of non-perturbative effects has been advocated for a long time [23]. To illustrate this idea, consider the one-loop formula for the running coupling $\alpha_s(p)$:

$$\alpha_s(p^2) = \frac{1}{b_0 \log(p^2/\Lambda^2)}. \tag{3.6}$$

[6] Note that higher powers would be subleading both for renormalons and for power corrections.

Here the singularity is a simple pole, which can be removed if one redefines $\alpha_s(p)$ according to the following prescription:

$$\alpha_s(p^2) = \frac{1}{b_0 \log(p^2/\Lambda^2)} + \frac{\Lambda^2}{b_0(\Lambda^2 - p^2)}\,, \tag{3.7}$$

where a power correction of the asymptotic form Λ^2/p^2 appears. However, the sign of such a correction is the opposite of what one would expect from the results of [8] and from the results in section 2 (although the absolute value is of the right order), so that in the end one could envisage a more general formula for the regularized coupling:

$$\alpha_s(p^2) = \frac{1}{b_0 \log(p^2/\Lambda^2)} + \frac{\Lambda^2}{b_0(\Lambda^2 - p^2)} + \eta\frac{\Lambda^2}{p^2}\,. \tag{3.8}$$

The message from (3.8) is that the coefficient of the power correction is not constrained by the mere cancellation of the pole.

At higher perturbative orders one encounters multiple singularities, which include an unphysical cut. There are several ways to regularize them. In particular, the method discussed in [23] combines a spectral-representation approach with the Renormalization Group. The method was originally formulated for QED, but it has recently been extended to the QCD case [24].

4. Conclusions

We have studied the strong coupling constant estimated non perturbatively on the lattice from Green functions in the Landau gauge using the $\widetilde{\mathrm{MOM}}$ scheme. This has been performed with a large statistics of 1000 field configurations per run and running at $\beta = 6.0, 6.2, 6.4, 6.8$. Finite volume effects as well as finite lattice spacing effects have been carefully controlled.

We have parametrized the momentum dependence of α_s using the three-loop perturbative formula plus a c/p^2 term. We have obtain a good fit in the energy range from 2.0 GeV up to 10.0 GeV. As a result of this study we find

$$\Lambda_{\overline{\mathrm{MS}}} = 237 \pm 3\,^{+\,0}_{-10}\,\mathrm{MeV}\,, \qquad c = 0.63 \pm 0.03\,^{+\,0.0}_{-\,0.13}\,\mathrm{GeV}^2\,, \tag{4.1}$$

$\Lambda_{\overline{\mathrm{MS}}}$ agrees perfectly well with the result of [4] and the existence of sizable power corrections is convincingly established. The stability of our fit has been extensively checked.[7]

The power correction turns out to be rather large, providing a 3% correction on α_s at 10.0 GeV, i.e. a 20% correction on Λ.

[7]Note added in proof: this result corresponds to the values of a^{-1}: 1.9657,2.75,3.66,6.1 GeV for β = 6.0, 6.2, 6.4, 6.8 respectively.

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Having gathered evidences for a particular scheme, one needs to address the issue of the general relevance of such a finding in the spirit of the discussion of section 3, assessing the scheme dependence of our results. As already discussed, the non-perturbative nature of power corrections makes it very hard to formulate any theoretical procedure to estimate the impact of scheme dependence. The best one can do at this stage is to consider different renormalization schemes and definitions of the coupling and gather numerical evidence and formal arguments supporting power corrections to $\alpha_s(p)$. In this way, scheme-independent features may eventually be identified. For example, on the basis of our results, we note the following:

- Theoretical arguments suggest $1/p^2$ corrections both for the coupling as defined from the static potential and for the one obtained from the three-gluon vertex. The arguments for the former case were outlined in sections 3.1 and 3.2. As far as the coupling from the three-gluon vertex is concerned, $1/p^2$ corrections appear in an OPE analysis if one keeps into account the fact that for such a gauge dependent coupling dimension 2 condensates are expected.

- In the static potential case, the theoretical arguments also provide an estimate for the order of magnitude of the coefficient of the $1/p^2$ correction while in the three-gluon vertex case the OPE arguments do not, suggesting instead that it may depend on the gauge. Nevertheless , the order of magnitude of our numerical result in the Landau gauge is roughly the same as the one from the static potential case. This calls for further investigation, which may be performed for example by attempting a similar calculation in a different gauge and particularly in the Coulomb gauge in which the static potential is naturally defined.

This issue of scheme dependence could be the focus of possible future work.

Acknowledgments

We thank Alain Le Yaouanc and Chris Michael for stimulating discussions. J. R-Q is indebted to Spanish Fundación Ramón Areces for his financial support. F. DR acknowledges both support from PPARC and from MURST (contract 9702213582) and INFN (i.s. PR11). C. Pi. warmly thanks the "Groupe de Phys. Nucl. Th. de l'Univ. de Liège" for kind hospitality and acknowledges the partial support of IISN. These calculations were performed on the QUADRICS QH1 located in the Centre de Ressources Informatiques (Paris-sud, Orsay) and purchased thanks to a funding from the Ministère de l'Education Nationale and the CNRS.

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